DIRECTLY INTERACTING MASSLESS PARTICLES - A TWISTOR APPROACH [1].

Andreas Bette,
Stockholm University,
Department of Physics,
Box 6730,
S-113 85 Stockholm, Sweden.
tfx: +468-347817 att. Andreas Bette,
e-mail: <ab@vanosf.physto.se>

Abstract

Twistor phase spaces are used to provide a general description of the dynamics of a finite number of directly interacting massless spinning particles forming a closed relativistic massive and spinning system with an internal structure.

A Poincaré invariant canonical quantization of the so obtained twistor phase space dynamics is performed.

1 INTRODUCTION.

It is possible that "elementary" massive particles such as electron, proton, neutron etc. should be regarded as bound states of a finite number of massless and spinning interacting parts.

In order to investigate such a possibility we develop in this paper a general formalism with its roots in the Twistor Theory of Penrose[1] and in the Theory of Action at a Distance in Relativistic Particle Dynamics (in its instantaneous form)[2].

Somewhat similar attempts to classify elementary particles employing the Twistor Theory but without any explicit mention of the Theory of Action at a Distance in Relativistic Particle Dynamics (R.a-a-a-d), have been made before by Hughston[3] and Popovich[4]. Certain other developments in the same direction appeared in papers written by Perjès et. al.[5, 6] and Sparling[7].

The quantum version of R.a-a-a-d in connection with the Twistor Theory seems to be implicit in an example worked out by Hughston[8].

The framework we are presenting is however from the very beginning completely in accordance with R.a-a-a-d. Classical states, which correspond to relativistic quantum bound states, of a massive and spinning composite particle, are represented by points in a finite dimensional "twistor phase space". The description is purely hamiltonian. The suggested quantization procedure is simultaneously canonical and Poincaré invariant. The arising canonically conjugated quantum mechanical operators represent "square roots" of the null-momenta and "square roots" of the "positions" attributed to (the classical limit of) the massless constituents forming a massive system.

Exploring the idea of instantaneous relativistic action at a distance[2] in the phase space of two twistors we have shown previously[9] how a free massive, spinning point-like particle may be thought of as a relativistic rigid rotator (endowed with intrinsic spin) composed of two massless spinning parts. The term "instantaneous" refers to the rest frame defined by the total time-like four-momentum of the rigid rotator itself. The present paper may be regarded as an extension and generalisation of the same idea. In its very rough state the idea appeared in our report[10] from 1979.

The work is organized as follows: First in the next section some results from the Twistor Theory are reformulated in a way which exhibits how they tie in with a relativistic, classical, finite dimensional phase space mechanics of a massless particle with helicity. Ten Poincaré covariant functions fulfilling Poisson bracket algebra of the Poincaré group and a function corresponding to the helicity operator are identified[1].

These well-known results in new clothes are generalised in section three where a twistor phase space of a massive spinning system composed of an arbitrary finite number of massless parts is introduced.

Again ten Poincaré covariant functions fulfilling Poisson bracket algebra of the Poincaré group are identified. Four functions representing the (real) position four-vector of the total massive spinning system are also identified[3].

A fundamental set of Poincaré scalar functions forming a closed finite Poisson bracket subalgebra is identified. These functions serve as generators (eigenvalues of their quan-

electric charge, baryonnumber etc.) of the internal symmetries.

A general formula for the four functions representing the Pauli-Lubański spin four-vector is derived.

A general formula for the function representing the square of the relativistic spin in terms of the generators of the internal symmetries is found.

It is discovered that, provided the relativistic spin does not vanish, the components of the position four-vector of the total system do not Poisson commute (see (3.27)).

A certain class of Poincaré scalar functions in the twistor phase space is selected. Functions in this class, when used as generators of motion, produce canonical flows which in Minkowski space describe a finite number of mutually interacting spinning massless particles forming a closed freely moving massive and, in general, spinning relativistic system.

The so obtained relativistic dynamics constitutes our general dynamical principle.

In the last section this classical phase space dynamics is canonically quantized in a way which corresponds to the real polarization of the twistor phase space[11]. A Poincaré invariant scalar product, on the space of functions representing quantum states of the massive spinning system composed of a finite number of massless parts, is introduced.

The following notation will be used:

Latin letters with lower case latin indices will denote four-vectors and four-tensors. Lower case greek letters with upper case latin indices (either primed or unprimed) will denote spinors. Upper case latin letters with lower case greek indices will denote non-projective twistors. Lower case latin indices within round brackets are used to number the different massless parts and in this way label the internal degrees of freedom. A bar over a letter or over an expression denotes complex conjugation. The usual summation convention over repeted indices is assumed throughout. The physical units are so chosen that $c = \hbar = 1$. The signature of the metric g_{ij} in Minkowski space is taken to be $+---$. The fully antisymmetric alternating four-tensor will be denoted by ϵ_{ijkl}.

2 THE ELEMENTARY TWISTOR PHASE SPACE.

There are no new results in this section except from the way they are presented.

We introduce the notion of a twistor phase space **Tp** which will be regarded as a space of classical states arising as a limit of some corresponding (not yet specified) quantum mechanical description of a massless particle with helicity. The value of such a quantum mechanical helicity is supposed to arise as an eigenvalue of some appropriate (not yet specified) quantum mechanical helicity operator. In the classical limit this quantum mechanical helicity operator should then correspond to a real valued function on **Tp.** Therefore the classical helicity (being a limit of a quantum mechanical operator whose eigenvalues are discrete) is not a discrete variable. In addition, following Penrose[1] and Hughston[3], ten real valued Poincaré covariant functions (corresponding to the generators of the Poincaré algebra) on **Tp** are identified as (classical) physical observables.

DEFINTION 1

A non-projective twistor space $\mathbf{T}$ is a four dimensional complex vector space (i.e. $C^4 \cong R^8$) endowed with the isometry group $SU(2,2)$.

REMARK 1:

$SU(2,2)$ is to be identified with (the universal covering of) the so called conformal group of the compactified Minkowski space and it contains as one of its subgroups (the universal covering of) the Poincaré group which, in turn, contains as one of its subgroups (the universal covering of) the Lorentz group i.e. $SL(2,C)$. As well-known the Poincaré group is the isometry group of the physical Minkowski space.

In order to see how vectors in $\mathbf{T}$ are related to physical Minkowskian quantities (such as angular and linear four-momenta, position four-vectors and Poincaré invariant scalars) it is convenient to choose a basis in $\mathbf{T}$ in a very special way. A vector in $\mathbf{T}$ given with respect to such a basis is called a non-projective twistor. With respect to any such a twistor basis the $SU(2,2)$ metric is non-diagonal.

DEFINTION 2:

A non-projective twistor Z^α and the corresponding (twistor) complex conjugated twistor $\bar{Z}_\alpha$ may thus be represented by two Weyl spinors and their conjugates:

$$Z^\alpha = (\omega^A,\ \pi_{A'}), \qquad \bar{Z}_\alpha = (\bar{\pi}_A,\ \bar{\omega}^{A'}). \tag{2.1}$$

REMARK 2:

Such a spinor representation of a non-projective twistor and its twistor conjugate also explicitly shows how the Poincaré group acts on $\mathbf{T}$. Coordinates of the two spinors represented by $\pi_{A'}$ and ω^A are covariant with respect to the (identity connected part of the) Lorentz group while four-translations T^a act only on the "ω" spinor parts of the twistor Z and its (twistor) complex conjugate[1] $\bar{Z}$.

DEFINTION 3:

The elementary twistor phase space $\mathbf{Tp}$ is spanned by R^8 in which each point is labelled by a non-projective twistor and its twistor complex conjugate. Further, $\mathbf{Tp}$ is equipped with an $SU(2,2)$ invariant symplectic structure[3, 12, 13] defined by the following canonical Poisson bracket relations:

$$\{Z^\alpha,\ \bar{Z}_\beta\} = i\delta^\alpha_\beta, \qquad \{Z^\alpha,\ Z^\beta\} = \{\bar{Z}_\alpha,\ \bar{Z}_\beta\} = 0, \tag{2.2}$$

which, when written out in terms of spinors, reads:

$$\{\omega^A,\ \bar{\pi}_B\} = i\delta^A_B, \qquad \{\pi_{B'},\ \bar{\omega}^{A'}\} = i\delta^{A'}_{B'}, \tag{2.3}$$

$$\{\omega^A,\ \omega^B\} = \{\omega^A,\ \pi_{A'}\} = \{\pi_{A'},\ \pi_{B'}\} = \{\pi_{A'},\ \bar{\pi}_B\} = 0, \tag{2.4}$$

$$\{\bar{\omega}^{A'},\ \bar{\omega}^{B'}\} = \{\bar{\omega}^{A'},\ \bar{\pi}_A\} = \{\bar{\pi}_A,\ \bar{\pi}_B\} = \{\omega^A,\ \bar{\omega}^{B'}\} = 0. \tag{2.5}$$

REMARK 3:

Points in **Tp** represent classical states of (the classical limit of) a massless particle with (any value of its) helicity.

LEMMA 1:

If the linear four-momentum P_a and the angular four-momentum $M_{ab} = -M_{ba}$ of a massless particle (Penrose's abstract index notation[14] is used throughout the paper when appropriate) are defined[1] by the following set of Poincaré covariant functions on **Tp:**

$$P_a := \pi_{A'}\bar{\pi}_A, \tag{2.6}$$

$$M_{ab} := i\bar{\omega}_{(A'}\pi_{B')}\epsilon_{AB} - i\omega_{(A}\bar{\pi}_{B)}\epsilon_{A'B'}, \tag{2.7}$$

then the canonical Poisson brackets (2.3)-(2.5) imply that P_a and M_{ab} fulfil the Poisson bracket relations of the Poincaré algebra[3]:

$$\{P_a,\ P_b\} = 0, \tag{2.8}$$

$$\{M_{ab},\ P_c\} = 2g_{c[a}P_{b]}, \tag{2.9}$$

$$\{M_{ab},\ M_{cd}\} = 2(g_{c[a}M_{b]d}) + g_{d[b}M_{a]c}). \tag{2.10}$$

Proof, which we omit, is just a tedious but straightforward computation. Penrose's blob notation[15] may be useful for this.

REMARK 4:

The above Poisson bracket relations define the momentum mapping for the action of the Poincaré group on **Tp.**

REMARK 5:

A point in **Tp** carries more information about the classical state of a massless particle than just information about its linear and angular four-momenta. It also defines its helicity and its phase.

LEMMA 2:

The helicity (state) function is given by[1]:

$$s = \frac{1}{2}(Z^\alpha \bar{Z}_\alpha) = \frac{1}{2}(\omega^A\bar{\pi}_A + \pi_{A'}\bar{\omega}^{A'}), \tag{2.11}$$

which may be easily deduced if in the definition of the Pauli-Lubański spin four-vector:

$$S^a := \frac{1}{2}\epsilon^{abcd}P_bM_{cd} \tag{2.12}$$

the expressions in (2.6)-(2.7) and the spinor version of ϵ^{abcd} are inserted. The result in (2.11) follows from a simple spinor algebra calculation[13] which yields:

$$S^a = sP^a. \tag{2.13}$$

REMARK 6:

Note that the massless particle's helicity function s coincides with one half of the $SU(2,2)$ norm of the corresponding non-projective twistor.

3 THE GENERAL TWISTOR PHASE SPACE.

A generalisation of the results presented in the previous section opens some new ways for applications of the Twistor Theory and of the R.a.a.a.

Namely, it becomes possible to formulate a general dynamical principle which, according to our interpretations and identifications, describes a closed massive and, in general, spinning system composed of a finite number of mutually interacting massless and spinning parts.

A direct product of any number of **Tp** may be used to define a (reducible) phase space for a massive spinning relativistic particle built up out of the massless ones. In such a direct product **Tp(n)** of n ($n \geq 2$) copies of the elementary twistor phase space **Tp** we exclude all points on all diagonals i.e. each point in **Tp(n)** represents a state of n massless particles with their four-momenta pointing at n non-coinciding null-directions.

Generalising the definition 3 of the previous section we let the symplectic structure on the product twistor phase space **Tp(n)** be given by the following set of canonical conformally invariant Poisson brackets:

DEFINTION 4:

$$\{Z^\alpha_{(i)},\ \bar{Z}_{\beta(j)}\} = i\delta^\alpha_\beta \delta_{(i)(j)}, \quad \{Z^\alpha_{(i)},\ Z^\beta_{(j)}\} = \{\bar{Z}_{\alpha(i)},\ \bar{Z}_{\beta(j)}\} = 0, \quad (i),(j) = 1,2..,n \tag{3.1}$$

where the index within brackets labels the n distinct massless parts.

LEMMA 3:

If the linear and angular four-momenta functions of the massive and spinning particle, formed by the n massless spinning constituents, are defined by:

$$\mathcal{P}_a := \pi_{A'(i)}\bar{\pi}_{A(i)}, \tag{3.2}$$

$$\mathcal{M}_{ab} := i\bar{\omega}_{(j)(A'}\pi_{B')(j)}\epsilon_{AB} - i\omega_{(j)(A}\bar{\pi}_{B)(j)}\epsilon_{A'B'}, \tag{3.3}$$

then the canonical commutation relations in (3.1) imply:

$$\{\mathcal{P}_a,\ \mathcal{P}_b\} = 0, \tag{3.4}$$

$$\{\mathcal{M}_{ab},\ \mathcal{P}_c\} = 2g_{c[a}\mathcal{P}_{b]}, \tag{3.5}$$

which, as should be expected, again represents Poisson bracket algebra of the Poincaré group.

REMARK 7:

Using the canonical (conformally covariant) twistor coordinates (on the $8n$ (real) dimensional twistor phase space) we note that they may be used to form $2n^2 - n$ real valued Poincaré scalar functions. n^2 of these are also conformally (i.e. $SU(2,2)$) invariant.

DEFINITION 5:

The n^2 real valued $SU(2,2)$ invariant scalars are represented by real and imaginary parts of the following functions:

$$a_{(i)(j)} := Z^{\alpha}_{(i)}\bar{Z}_{\alpha(j)}, \qquad \bar{a}_{(i)(j)} = a_{(j)(i)} \tag{3.7}$$

while the remaining Poincaré invariant scalars are represented by real and imaginary parts of:

$$m_{(i)(j)} := I_{\alpha\beta}Z^{\alpha}_{(i)}Z^{\beta}_{(j)} = \epsilon^{C'D'}\pi_{D'(i)}\pi_{C'(j)} = -m_{(j)(i)} \tag{3.8}$$

$$\bar{m}_{(i)(j)} := I^{\alpha\beta}\bar{Z}_{\alpha(i)}\bar{Z}_{\beta(j)} = \epsilon^{CD}\bar{\pi}_{D(i)}\bar{\pi}_{C(j)} = -\bar{m}_{(j)(i)} \tag{3.9}$$

where ϵ^{AB}, $\epsilon^{A'B'}$ denote the metric in the Weyl spinor space or equivalently $I^{\alpha\beta}$ and $I_{\alpha\beta}$ denote the so called infinity twistor and its dual[1, 13].

LEMMA 4:

From the canonical commutation relations in (3.1) it almost trivially follows that:

$$\{a_{(i)(j)},\ Z^{\alpha}_{(k)}\} = -i\delta_{(j)(k)}Z^{\alpha}_{(i)} \qquad \{a_{(i)(j)},\ \bar{Z}_{\alpha(k)}\} = i\delta_{(i)(k)}\bar{Z}_{\alpha(j)} \tag{3.10}$$

$$\{m_{(i)(j)},\ Z^{\alpha}_{(k)}\} = 0 \qquad \{\bar{m}_{(i)(j)},\ Z^{\alpha}_{(k)}\} = 2iI^{\alpha\mu}\bar{Z}_{\mu[(i)}\delta_{(j)](k)} \tag{3.11}$$

$$\{m_{(i)(j)},\ \bar{Z}_{\alpha(k)}\} = 2iI_{\mu\alpha}Z^{\mu}_{[(i)}\delta_{(j)](k)} \qquad \{\bar{m}_{(i)(j)},\ \bar{Z}_{\alpha(k)}\} = 0 \tag{3.12}$$

LEMMA 5:

As shown by Hughston[3] the real four-vector valued function on $\mathbf{Tp(n)}$ representing, in the Minkowski space, position four-vector of the total system (composed of n massless parts) is given by:

$$X^a = X^{AA'} := \frac{\mathcal{M}^{ab}\mathcal{P}_b}{m^2} + \frac{l}{m^2}\mathcal{P}^a \tag{3.13}$$

where

$$l := -\frac{1}{2}(i\omega^{A}_{(i)}\bar{\pi}_{A(i)} - i\pi_{A'(i)}\bar{\omega}^{A'}_{(i)}), \tag{3.14}$$

and where

$$m^2 := \mathcal{P}_a \mathcal{P}^a = m_{(i)(j)} \bar{m}_{(i)(j)} \tag{3.15}$$

or equivalently by[3]:

$$X^a = i \frac{1}{m^2} [\bar{m}_{(i)(j)} \omega^A_{(i)} \pi^{A'}_{(j)} - m_{(i)(j)} \bar{\omega}^{A'}_{(i)} \bar{\pi}^A_{(j)}]. \tag{3.16}$$

Now it is a straightforward task to calculate the following Poincaré invariant and Poincaré covariant Poisson bracket commutation relations which will be needed in the sequel.

LEMMA 6:

First we note that from the conformally invariant canonical Poisson bracket relations (3.1) it follows that the $2n^2 - n$ scalars in (3.7)-(3.9) form a Poincaré invariant closed algebra of Poisson brackets:

$$\{a_{(i)(j)},\ a_{(k)(l)}\} = ia_{(k)(j)} \delta_{(i)(l)} - ia_{(i)(l)} \delta_{(j)(k)}, \quad \{a_{(i)(j)},\ m_{(k)(l)}\} = 2im_{(i)[(k)} \delta_{(l)](j)} \tag{3.17}$$

$$\{a_{(i)(j)},\ \bar{m}_{(k)(l)}\} = 2i\bar{m}_{(j)[(k)} \delta_{(l)](i)}, \quad \{m_{(i)(j)},\ m_{(k)(l)}\} = \{\bar{m}_{(i)(j)},\ m_{(k)(l)}\} = 0. \tag{3.18}$$

which may easily be proved by the help of lemma 4.

LEMMA 7:

From the fact that $m_{(i)(j)}$, $\bar{m}_{(i)(j)}$ and $a_{(i)(j)}$ are Poincaré scalar functions it trivially follows that they commute with all the generators of the Poincaré algebra:

$$\{a_{(i)(j)},\ \mathcal{P}_a\} = \{a_{(i)(j)},\ \mathcal{M}_{ab}\} = 0 \tag{3.19}$$

$$\{m_{(i)(j)},\ \mathcal{P}_a\} = \{m_{(i)(j)},\ \mathcal{M}_{ab}\} = \{\bar{m}_{(i)(j)},\ \mathcal{P}_a\} = \{\bar{m}_{(i)(j)},\ \mathcal{M}_{ab}\} = 0. \tag{3.20}$$

LEMMA 8:

The following commutation relations are also easily deduced from the canonical commutations relations in (3.1):

$$\{a_{(i)(j)}, l\} = 0 \tag{3.21}$$

$$\{m_{(i)(j)},\ l\} = m_{(i)(j)}. \tag{3.22}$$

LEMMA 9:

From (3.13) and (3.17)-(3.22) it now follows that:

$$\{m_{(i)(j)},\ X^a\} = \frac{1}{m^2}\mathcal{P}^a\{m_{(i)(j)},\ l\} = \frac{1}{m^2}\mathcal{P}^a m_{(i)(j)} \tag{3.24}$$

$$\{\mathcal{P}_b,\ X^a\} = \delta^a_b \tag{3.25}$$

$$\{\frac{1}{2}\mathcal{P}_b\mathcal{P}^b,\ X^a\} = \mathcal{P}^a. \tag{3.26}$$

LEMMA 10:

Similarily we obtain that:

$$\{X^a,\ X^b\} = \frac{1}{m^4}\epsilon^{abcd}\mathcal{S}_c\mathcal{P}_d \tag{3.27}$$

where the Pauli-Lubański four-vector $\mathcal{S}^a$ is defined as in (2.12) i.e. by:

$$\mathcal{S}^a := \frac{1}{2}\epsilon^{abcd}\mathcal{P}_b\mathcal{M}_{cd}. \tag{3.28}$$

LEMMA 11:

Expressing the right hand side of (3.28) in terms of spinors defining the corresponding twistors we obtain:

$$2\mathcal{S}^a = 2\mathcal{S}^{AA'} = \bar{m}_{(i)(j)}\omega^A_{(i)}\pi^{A'}_{(j)} + m_{(i)(j)}\bar{\omega}^{A'}_{(i)}\bar{\pi}^A_{(j)} + a_{(i)(j)}\bar{\pi}^A_{(i)}\pi^{A'}_{(j)} \tag{3.29}$$

which after some spinor algebra manipulations may be rewritten as:

$$2\mathcal{S}^{AA'} = [2a_{(i)(j)} - \delta_{(i)(j)}a_{(k)(k)}]\bar{\pi}^A_{(i)}\pi^{A'}_{(j)} = 2a_{(i)(j)}\bar{\pi}^A_{(i)}\pi^{A'}_{(j)} - a_{(i)(i)}\bar{\pi}^A_{(j)}\pi^{A'}_{(j)}. \tag{3.30}$$

REMARK 8:

The formula in (3.30) is also valid for $n = 1$ and reproduces the result[1] of lemma 2. For $n = 2$ it appeared in Tod's doctoral dissertation[18]. However, the author of this paper has never come across the general formula in (3.30) which is valid for any (finite) natural number n.

LEMMA 12:

From the above lemma (lemma 11) it follows that the square of the value of the total spin s^2 (for $n > 2$) is a function of the invariants in (3.7)-(3.9) (i.e. is a function of the generators of the internal symmetries) given by:

$$-4m^2s^2 := 4\mathcal{S}^a\mathcal{S}_a =$$
$$(a_{(j)(j)})^2m^2 + 4a_{(j)(j)}a_{(u)(v)}\bar{m}_{(u)(k)}m_{(k)(v)} + 4a_{(j)(k)}a_{(u)(v)}\bar{m}_{(j)(u)}m_{(k)(v)}. \tag{3.31}$$

REMARK 9:

For $n = 2$ and $n = 3$ the formula (3.31) agrees with those previously derived by Perjès, Hughston, Sparling. The general formula above is, however, valid for any (finite) natural

PROPOSITION 1:

As explained in the introduction we wish to regard a closed massive and spinning system as composed of a finite number of interacting massless parts. For this reason we notice that any function of the form

$$H := \frac{1}{2}\mathcal{P}_b\mathcal{P}^b + g(a_{(j)(k)},\ m_{(l)(m)},\ \bar{m}_{(n)(r)}), \tag{3.32}$$

where g is a positive real valued function of the invariants in (3.7)-(3.9), generates a canonical flow in $\mathbf{Tp(n)}$ which in the Minkowski space describes a set of n mutually interacting massless particles. This follows from direct calculations which produce the following equations of the motion:

$$\dot{X}^a = \{H,\ X^a\} = \mathcal{P}^a(1 + \frac{1}{m^2}\frac{\partial g}{\partial m^{(i)(k)}}m_{(i)(k)} + \frac{1}{m^2}\frac{\partial g}{\partial \bar{m}^{(i)(k)}}\bar{m}_{(i)(k)}), \tag{3.33}$$

$$\dot{\pi}_{A'(k)} = \{H,\ \pi_{A'(k)}\} = -i\frac{\partial g}{\partial a^{(j)(k)}}\pi_{A'(j)}, \qquad \dot{\mathcal{P}}^a = \{H,\ \mathcal{P}^a\} = 0, \tag{3.34}$$

$$\dot{a}_{(k)(j)} = \{H,\ a_{(k)(j)}\} =$$

$$i\frac{\partial g}{\partial a^{(j)(l)}}a_{(k)(l)} - i\frac{\partial g}{\partial a^{(l)(k)}}a_{(l)(j)} - 2i\frac{\partial g}{\partial \bar{m}^{(k)(l)}}\bar{m}_{(l)(j)} + 2i\frac{\partial g}{\partial m^{(j)(l)}}m_{(k)(l)}. \tag{3.35}$$

REMARK 10:

Note that the assumption in (3.32), stating that g is a function of the generators $a_{(i)(j)}$ (which are conformal scalars), makes the motion of the massless parts non-trivial (i.e. changes their null-momenta during the motion). All functions g, which depend on $m_{(i)(j)}$ and their complex conjugates only, produce a motion of the massless parts which is trivial in the Minkowski space.

ASSUMPTION 1:

From now on we assume that g is a function of the conformal invariants $a_{(i)(j)}$ only:

$$g = g(a_{(j)(k)}). \tag{3.36}$$

PROPOSITION 2:

Under this condition the equations of motion generated by the canonical flow in the twistor phase space simplify and read:

$$\dot{X}^a = \{H,\ X^a\} = \mathcal{P}^a, \tag{3.37}$$

$$\dot{\pi}_{A'(k)} = \{H,\ \pi_{A'(k)}\} = -i\frac{\partial g}{\partial a^{(j)(k)}}\pi_{A'(j)}, \qquad \dot{\mathcal{P}}^a = \{H,\ \mathcal{P}^a\} = 0, \tag{3.38}$$

$$\dot{a}_{(k)(j)} = \{H,\ a_{(k)(j)}\} = i\frac{\partial g}{\partial a^{(j)(l)}}a_{(k)(l)} - i\frac{\partial g}{\partial a^{(l)(k)}}a_{(l)(j)}. \tag{3.39}$$

REMARK 11:

From (3.37) it follows that the parameter labelling points on the curves of the canonical flow generated by H with g such as in (3.36) is linearly related to the proper time of the total system.

If such a function g vanishes (or degenerates to a real number) then the function H and the function $\frac{1}{2}m^2$ are identical (modulo an additive real number) forming just one constant of the free motion generated by H.

For non-trivial such g the functions H and $\frac{1}{2}m^2$ correspond to two different mutually commuting but à priori unrelated constants of the motion generated by H.

REMARK 12:

The above equations of motion have been explicitly solved[9] for $n = 2$ and for $g = s^2$. Such a motion describes a massive relativistic rigid rotator composed of two directly interacting massless spinning particles.

ASSUMPTION 2:

Due to the fact that the parameter labelling the curves of the canonical flow generated by H is linearly related to the proper time of the total system, we make an additional "physical" assumption that for any non-trivial g such as in (3.36) the constant of the motion given by the value of the function H is proportional (modulo an additive real number r) to the value of the constant of the motion $\frac{1}{2}m^2$. The proportionality constant k is larger than one half and aproaches one half when the function g approaches zero (modulo an additive real number r):

$$H = km^2 + r \qquad k > \frac{1}{2} \tag{3.40}$$

À posteriori this amounts to a "constraint":

$$m^2 = \frac{g - r}{(k - \frac{1}{2})} \tag{3.41}$$

The imposition of such a "constraint" seems perhaps somewhat unnecessary at this stage but may be motivated by the fact that after quantization we wish to interpret ratios of the arising possibly discrete eigenvalues of $\hat{H}$ (for some specific choices of g in (3.36)) as ratios of the squares of the quantized masses.

4 QUANTIZATION.

A non-standard[11], as opposed to the standard procedure introduced by Penrose[1], canonical twistor quantization is obtained by means of a natural prescription á la Dirac[16, 17] given by:

$$\hat{\omega}^A_{(i)} := -\frac{\partial}{\partial \bar{\pi}^{(i)}_A}, \qquad \hat{\bar{\omega}}^{A'}_{(i)} := \frac{\partial}{\partial \pi^{(i)}_{A'}}, \tag{4.1}$$

$$\hat{\bar{\pi}}_{A(i)} := \bar{\pi}_{A(i)}, \qquad \hat{\pi}_{A'(i)} := \pi_{A'(i)}. \tag{4.2}$$

The Poisson brackets relations in (3.1) will hereby be replaced by the corresponding commutators turning the classical twistor phase space dynamics of massless particles

So by the use of (4.1)-(4.2) the linear four-momentum functions in (3.2), the angular four-momentum functions in (3.3), the scalar functions in (3.7)-(3.9) turn into the corresponding operators:

$$\hat{\mathcal{P}}_a := \bar{\pi}_{A(i)}\pi_{A'(i)}, \tag{4.3}$$

$$\hat{\mathcal{M}}^{ab} := i\pi^{(A'}_{(i)}\frac{\partial}{\partial\pi^{(i)}_{B')}}\epsilon^{AB} + i\bar{\pi}^{(A}_{(i)}\frac{\partial}{\partial\bar{\pi}^{(i)}_{B)}}\epsilon^{A'B'}, \tag{4.4}$$

$$\hat{a}_{(i)(j)} := -\bar{\pi}_{A(j)}\frac{\partial}{\partial\bar{\pi}^{(i)}_{A}} + \pi_{A'(i)}\frac{\partial}{\partial\pi^{(j)}_{A'}}, \quad \hat{m}_{(i)(j)} := \pi^{A'}_{(i)}\pi_{A'(j)}, \quad \hat{\bar{m}}_{(i)(j)} := \bar{\pi}^{A}_{(i)}\bar{\pi}_{A(j)}. \tag{4.5}$$

The Poisson bracket relations in (3.4)-(3.6) ensure that operators in (4.3) and (4.4) obey commutation relations of the Poincaré algebra. In addition all the Poisson bracket commutation relations in (3.10) - (3.12), (3.17) - (3.27) turn into the corresponding operator commutators.

All functions on **Tp(n)** become (at least formally) operator valued functions of the canonical differential operators in (4.1) and of the multiplicative operators in (4.2). Of course this may lead to problems: Ordering problems, non-locality of the operators arising from functions on **Tp(n)** in which the "ω" parts appear in the denominator etc.

The multiplicative operators in (4.2) define a Poincaré invariant $4n$ real dimensional configuration vector space Π spanned by n Weyl spinors and their complex conjugates.

The above (formally) defined operator valued functions of the canonical operators in (4.1)-(4.2) act on the infinitely dimensional space Γ of complex valued "wave" functions defined on Π.

A Poincaré invariant scalar product on the space of complex valued functions on Π we tentatively define as:

$$< f_1|f_2 >:= \int [\bar{f}_1(\bar{\pi}_{B(i)},\ \pi_{B'(i)})f_2(\pi_{B'(i)},\ \bar{\pi}_{B(i)})]d\pi^{A'}_{(j)} \wedge d\pi_{A'(j)} \wedge d\bar{\pi}^{A}_{(k)} \wedge d\bar{\pi}_{A(k)}. \tag{4.6}$$

The subspace $\aleph$ of Γ consisting of functions having finite norms with respect to this scalar product defines a Hilbert space of quantum states of the massive spinning composite particle.

The quantized version of our general dynamical principle now reads:

Find common eigenvalues and eigenfunctions of a maximal set of hermitian mutually commuting operators containing the following subset (N refers to the normal ordering of terms):

$$\hat{\mathcal{P}}_a = \bar{\pi}_{A(i)}\pi_{A'(i)}, \tag{4.7}$$

$$\hat{H} = \frac{1}{2}\hat{m}^2 + N(g(\hat{a}_{(i)(j)})) \tag{4.8}$$

$$S_z = -\frac{1}{m}\hat{\mathcal{S}}^a n_a = -\frac{1}{m}\hat{\mathcal{S}}^{AA'} n_{AA'} =$$

$$\frac{1}{2}(\hat{a}_{(i)(j)}\bar{\pi}^A_{(i)}\pi^{A'}_{(j)} + \bar{\pi}^A_{(i)}\pi^{A'}_{(j)}\hat{a}_{(i)(j)})n_{AA'} - \frac{1}{4}(\hat{a}_{(i)(i)}\bar{\pi}^A_{(j)}\pi^{A'}_{(j)} + \bar{\pi}^A_{(j)}\pi^{A'}_{(j)}\hat{a}_{(i)(i)})n_{AA'}. \quad (4.9)$$

$$\hat{m}^2\hat{s}^2 := -\hat{\mathcal{S}}^a\hat{\mathcal{S}}_a =$$

$$-\frac{1}{4}(\hat{a}_{(j)(j)})^2 m^2 - N(\hat{a}_{(j)(j)}\hat{a}_{(u)(v)}\bar{m}_{(u)(k)}m_{(k)(v)}) - N(\hat{a}_{(j)(k)}\hat{a}_{(u)(v)}\bar{m}_{(j)(u)}m_{(k)(v)}). \quad (4.10)$$

where m denotes an eigenvalue of the mass and n_a is any space-like unit four-vector orthogonal to the time-like direction defined by $\hat{\mathcal{P}}_a$. In other words n^a represents a "z-axis" direction.

From the assumption 2 and general group theoretical considerations it follows that eigenvalues of the mass squared $\hat{m}^2$ are proportional (up to an additive real number) to the eigenvalues of $N(g(\hat{a}_{(i)(j)}))$ (which is assumed to be hermitian). Moreover the eigenvalues of the square of the spin $\hat{s}^2$ and of $\hat{S}_z$ assume the usual values i.e. $j(j+1)$ and $j_z = -j, ..., j$ with j being a positive integral number or a positive half integral number.

For each choice of g one can choose among the operators in (4.5) a maximal set of mutually commuting ones which also commute with $N(g(\hat{a}_{(i)(j)}))$. The eigenvalues of these additional operators may be identified with the internal degrees of freedom of the total system. To each set of eigenvalues of the mutually commuting internal operators and to each set of the mutually commuting external operators' eigenvalues (the mass, j, j_z, total four-momentum of the system) there corresponds a state function in $\aleph$ which may be calculated (at least in principle) using methods from non-relativistic quantum mechanics.

As an example consider the relativistic rigid rotator composed of two ($n = 2$) massless constituents with helicity[9]. To quantize it we note that the external commuting observables may be chosen as:

$$\hat{\mathcal{P}}_a = \bar{\pi}_{A(1)}\pi_{A'(1)} + \bar{\pi}_{A(2)}\pi_{A'(2)}, \quad (4.11)$$

$$\hat{H} = \frac{1}{2}\hat{m}^2 + \hat{s}^2, \quad (4.12)$$

$$\hat{\mathcal{S}}^a n_a = \mathcal{S}^{AA'} n_{AA'} =$$

$$\frac{1}{4}(\hat{a}_{(1)(1)} - \hat{a}_{(2)(2)})(\bar{\pi}^A_{(1)}\pi^{A'}_{(1)} - \bar{\pi}^A_{(2)}\pi^{A'}_{(2)})n_{AA'} + (\bar{\pi}^A_{(1)}\pi^{A'}_{(1)} - \bar{\pi}^A_{(2)}\pi^{A'}_{(2)})n_{AA'}\frac{1}{4}(\hat{a}_{(1)(1)} - \hat{a}_{(2)(2)})$$

$$+\frac{1}{2}(\hat{a}_{(1)(2)}\bar{\pi}^A_{(1)}\pi^{A'}_{(2)} + \bar{\pi}^A_{(1)}\pi^{A'}_{(2)}\hat{a}_{(1)(2)})n_{AA'} + \frac{1}{2}(\hat{a}_{(2)(1)}\bar{\pi}^A_{(2)}\pi^{A'}_{(1)} + \bar{\pi}^A_{(2)}\pi^{A'}_{(1)}\hat{a}_{(2)(1)})n_{AA'}, \quad (4.13)$$

$$\hat{s}^2 = \frac{1}{4}(\hat{a}_{(1)(1)} - \hat{a}_{(2)(2)})^2 + \frac{1}{2}(\hat{a}_{(1)(2)}\hat{a}_{(2)(1)} + \hat{a}_{(2)(1)}\hat{a}_{(1)(2)}), \quad (4.14)$$

while internal symmetry operators are given e.g. by:

In the rigid rotator case the eigenvalues of the square of the mass are proportional to $j(j+1)$ i.e. proportional to the eigenvalues of the square of the spin. In addition the states (eigenfunctions) of the rigid rotator are labelled by the eigenvalues of the Euler operators in (4.15).

To find these relativistic rigid rotator eigenfunctions $f(\pi^{A'}_{(1)},\ \pi^{A'}_{(2)},\ \bar{\pi}^{A}_{(1)},\ \bar{\pi}^{A}_{(2)})$ in $\aleph$ is a much harder task and will not be pursuited in this paper.

5 CONCLUSIONS AND REMARKS.

If "elementary" particles such as e.g. electron, proton, neutron etc. may be regarded as bound states of a finite number of massless spinning parts then twistor theory combined with the idea of relativistic action at a distance provide a very powerful tool for construction of such models.

In such approaches, as shown in this paper, particle aspects of Penrose's twistor formalism should be emphasized as opposed to the standard treatments where field aspects are at the front.

The non-standard quantization procedure in (4.1) - (4.2) implies that we loose some of the results of conventional twistor theory such as the twistor description of massless free fields in terms of holomorphic sheaf cohomology, the scalar product on such fields, geometrization of the concept of positive frequency of the field and the relationship between conformal curvature and the twistor "position" (twistor variables) and "momentum" (complex conjugates of the twistor variables) operators[19].

What we gain is that the real dimension of the relativistic configuration space of massless spinning particles is one half of the real dimension of the configuration space obtained by means of the conventional holomorphic twistor quantization[1]. Further, the configuration space obtained in our paper may be given a clear physical interpretation. Wave functions on such a configuration space define quantum states in (the "square root" of) the linear four-momentum representation of the massless parts. However in our opinion the most important gain is the fact that using our formulation we are able to treat *interacting* massless spinning particles (not fields) forming a closed composite bound system.

To apply our ideas to concrete physical systems is, at the moment, hampered by the fact that there are, as yet, no indications in the model how the function g in (3.36) should be chosen.

Nevertheless, the general principle presented in this paper seems to comply with the Twistor Programme announced by Penrose[20].

Acknowledgment.

Certain features of the idea presented in this note have been previously described in a talk given at VII-th Conference on "Symmetry Methods in Physics" - Dubna, 10-16 July, 1995. The visit to this conference was partly supported by NFR - Sweden, contract number R-RA 04666-306.

References

1. R. Penrose and M.A.H. MacCallum, Physics report C6(4), 241 (1972).
2. E.H. Kerner (Editor), "The theory of action-at-a-distance in relativistic particle dynamics - reprint collection", Gordon and Breach, Science Publishers, Inc. (1972).
3. L.P. Hughston, "Twistors and Particles", Lectures notes in Physics No.97, Springer Verlag, Berlin, Heidelberg, New York. (1979).
4. A.S. Popovich, Doctoral thesis at Oxford's Mathematical Institute, (1980).
5. Z. Perjès, Phys. Rev. D, Vol.20, No. 8, (1979).
6. B. Lukács, Z. Perjès, Á. Sebestyén, E. T. Newman, J. Porter, J. Math. Phys., 23 (11), November 1982.
7. G. A. J. Sparling, "Theory of Massive Particles", Proc. Roy. Soc. Vol. 301. A 1458 pp. 27-74, (1981).
8. L.P. Hughston, "The Relativistic Oscillator", Proc. Roy. Soc. London, A82, 459-466, (1982).
9. A. Bette, J. Math. Phys. Vol. 25, (1984).
10. A. Bette, "Twistors, the metastructure of particles and space-time - an introduction", Usip Report 79-10, (1979).
11. N. Woodhouse, "Geometric Quantization" - Second edition in Oxford Mathematical Monographs series, Oxford University Press, Clarendon Press-Oxford, p.126 and p.193, (1992).
12. R. Penrose, in "Magic without magic: J. A. Wheeler a collection of essays in honor of his 60th birthday.", ed. J.R. Klauder, W.H. Freeman and Co., San Francisco, (1972).
13. R. Penrose, and W. Rindler, "Spinors and Space-Time", Cambridge Monographs on Mathematical Physics vol.1 and 2, Cambridge University Press, (1984).
14. R. Penrose, in "Batelle Rencontres" eds. C.M. de Witt, J.A. Wheeler Princeton University W.A. Benjamin inc. New York, Amsterdam pp. 135-149, (1968).
15. R. Penrose, "Application of Negative Dimensional Tensors" in "Combinatorial Mathematics", edited by D.J.A. Welsh (Academic Press, New York), (1971).
16. P.A.M. Dirac, Proc. Roy. Soc. London ser. A, 109, 642-653, (1925).
17. P.A.M. Dirac, "The Principles of Quantum Mechanics", Fourth edition, Oxford, Clarendon Press, (1958).
18. K. P. Tod, "Massive Spinning Particles and Twistor Theory"-Doctoral Dissertation at Mathematical Institute, University of Oxford, (1975).
19. R. Penrose, Int.J.Theor.Phys., Vol.1, No.1, 61-99. (1968).
20. R. Penrose, Rep.Math.Phys., Vol.12, p.65, (1977).